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The Question of Uniqueness for G. D. Birkhoff Interpolation Problems

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INTRODUCTION

In this paper we generalize some results of Pólya and Schoenberg on the question of when polynomial interpolation schemes of the type studied by G. D. Birkhoff yield unique solutions.

Necessary and sufficient conditions are given for the existence of nodes for which the interpolation polynomials are unique. Then, those systems for which unique interpolation exists for all nodes are characterized.

We conclude by showing some partial characterizations of polynomia interpolation when the nodes are restricted to be real and then when their order is required to remain fixed.

1. STATEMENT OF THE PROBLEM

The interpolation problem that we wish to consider in this paper was first studied by G. D. Birkhoff [3] and can be stated as follows. Let there be given positive integers k, n, and n-ordered pairs (i,j), where i, j are integers with $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant n-1$. Let $x_1, x_2, \ldots, x_k$ be distinct complex numbers and for each of the above (i,j), let y_i^j be a given complex number. Does there exist a polynomial $p(x)$ of degree less than n which satisfies, for each of these (i,j), $p^{(j)}(x_i) = y_i^j$, and, if so, is this polynomial unique?

In this paper, we are only interested in the uniqueness question, and so we can state our problem thus: If $p(x)$ solves the interpolation problem when all the numbers y_i^j are zero, is $p(x)$ identically zero?

In [5], I. J. Schoenberg introduced the concept of an n-incidence matrix. A matrix E is called an *n-incidence matrix* if

$$E = \|e_{ij}\|_{j=0,\ldots,n-1}^{i=1,\ldots,k}, \quad \text{where each } e_{ij} \text{ is 0 or 1 and } \sum_{(i,j)} e_{ij} = n.$$

Thus, E has k rows and n columns, and has exactly n nonzero entries. Now our problem can be restated as follows: given an n-incidence matrix with k rows, and given k distinct points $x_1, \ldots, x_k$ and a polynomial $p(x)$ in the class π_{n-1}

of all polynomials of degree $\leqslant n-1$, which satisfies $p^{(j)}(x_i)=0$ if $e_{ij}=1$, is $p(x)$ identically zero? Whether it is identically zero or not, we say that $p(x)$ *interpolates the matrix E at the nodes* $x_1, \ldots, x_k$. We call n the order of the interpolation problem.

For notational ease we make the following

DEFINITION 1.1. Two n-column matrices E and $\tilde{E}$ are said to be equivalent, if they have the same number of nonzero rows, and the nonzero rows of $\tilde{E}$ are a permutation of the nonzero rows of E.

The significance of such equivalence stems from the fact that zero rows in an incidence matrix have no effect on the interpolation problem and, since any ordering of the interpolating points is immaterial (except in Section 5), the ordering of the rows of the matrix is incidental to the problem. Thus, the matrices

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

all define the same interpolation problem and are equivalent.

As an example of our problem, consider again the matrix

$$E = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}.$$

Here we are asked to find the polynomials of degree less than or equal to 2 which have zeroes at x_1 and x_3 and whose first derivatives vanish at x_2. If x_2 is taken to be the midpoint between x_1 and x_3, then the polynomial $p(x) = (x-x_1)(x-x_3)$ interpolates E at the nodes x_1, x_2 and x_3. However, if x_2 is any other point, the only interpolating polynomial is the zero polynomial. Thus, the question of uniqueness sometimes depends on the choice of the nodes x_i.

DEFINITION 1.2. Given an n-incidence matrix E and distinct points $x_1, \ldots, x_k$, E is said to be *poised with respect to* $x_1, \ldots, x_k$ if the only polynomial in π_{n-1} which interpolates E at these points is the zero polynomial. E is said to be *conditionally poised* if there are (distinct) points $x_1, \ldots, x_k$ with respect to which E is poised. E is *poised* (*or unconditionally poised*) if it is poised with respect to all choices of distinct points $x_1, \ldots, x_k$.

In this and the following two sections, we shall characterize poised and conditionally poised matrices.

Our first result deals with the possibility of reducing the order of a given interpolation problem.

Let E_1 and E_2 be, respectively, n_1- and n_2-incidence matrices. If $n = n_1 + n_2$, we define a class $E_1 \oplus E_2$ of n-incidence matrices as follows: $E \in E_1 \oplus E_2$ if the matrix $\tilde{E}_1$ consisting of the first n_1 columns of E and the matrix $\tilde{E}_2$ consisting of the last $n - n_1 = n_2$ columns of E are equivalent to E_1 and E_2, respectively. For example, if

$$E_1 = \begin{Vmatrix} 1 & 0 \\ 1 & 0 \end{Vmatrix}$$

and

$$E_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix},$$

then the class $E_1 \oplus E_2$ consists of the matrices

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{Vmatrix}$$

and equivalent matrices.

THEOREM 1.1. *Let E_1, E_2 be, respectively, n_1- and n_2-incidence matrices, and let $n = n_1 + n_2$. If $E \in E_1 \oplus E_2$ is poised with respect to given points $x_1, \ldots, x_r$, then E_1 and E_2 must both be poised with respect to the same points. Conversely, if E_1 and E_2 are poised with respect to $x_1, \ldots, x_r$, then every $E \in E_1 \oplus E_2$ is poised with respect to $x_1, \ldots, x_r$.*

Proof. Suppose E is poised with respect to the points $x_1, \ldots, x_r$. Let $p(x) \in \pi_{n_1-1}$ be a polynomial which interpolates E_1 at the given points. Then $p(x)$ satisfies all the interpolation data of E up to the $(n_1 - 1)$st column. But $p^{(n_1)}(x) \equiv 0$ and, hence, $p(x)$ satisfies the interpolation data of E in columns n_1 to $n - 1$. Thus, $p(x) \equiv 0$, showing that E_1 is poised with respect to $x_1, \ldots, x_r$.

Now let $q(x) \in \pi_{n_2-1}$ be a polynomial which interpolates E_2 at $x_1, \ldots, x_r$. Let $\tilde{q}(x) \in \pi_n$ be a polynomial such that $\tilde{q}^{(n_1)}(x) \equiv q(x)$. Since E_1 is poised for the given points, there is a unique polynomial $p(x) \in \pi_{n_1-1}$ satisfying $p^{(j)}(x_i) = -\tilde{q}^{(j)}(x_i)$ for $e_{ij}^1 = 1$ $(E_1 = \|e_{ij}^1\|)$. Now the polynomial $p(x) + \tilde{q}(x)$ interpolates E at $x_1, \ldots, x_r$ and, hence, is identically zero. Thus, $q(x) = D^{(n_1)}(p(x) + \tilde{q}(x)) \equiv 0$ and E_2 is poised with respect to $x_1, \ldots, x_r$.

Conversely, suppose that E_1 and E_2 are poised with respect to the points $x_1, \ldots, x_r$. If $p(x) \in \pi_{n-1}$ is a polynomial which interpolates E at these points, then $p^{(n_1)}(x) \in \pi_{n_2-1}$ interpolates E_2 at the given points. Thus, $p^{(n_1)}(x) \equiv 0$ and $p(x)$ is in π_{n_1-1}. But $p(x)$ also interpolates E_1 at these points and, hence, must be identically zero. Therefore, E is poised with respect to the given points and the theorem is demonstrated.

LEMMA 1.1. *Let*

$$E = \|e_{ij}\|_{i=1,\ldots,k}^{j=0,\ldots,n-1}$$

be a conditionally poised matrix. Then the set of vectors $\mathbf{X} = (x_1, \ldots, x_k)$ *for which E is not poised is a closed, nowhere dense set in complex k-space.*

Proof. E is poised with respect to given $x_1, \ldots, x_k$ if, and only if,

$$P(x_1, \ldots, x_k) = \det |D^{(j)} x_i^t|_{e_{ij}=1}^{t=0,\ldots,n-1} \neq 0,$$

where $D^{(j)} x_i^t$ denotes the jth derivative of x^t at $x = x_i$.

Now, the set of points $\mathbf{X} = (x_1, x_2, \ldots, x_k)$ for which $P(x_1, \ldots, x_k) = 0$ is closed. Also, if this set contained an open sphere in complex k-space, then we would have $P(x_1, x_2, \ldots, x_n) \equiv 0$, which is impossible since E is conditionally poised. Thus, the set on which $P(x_1, \ldots, x_k)$ is zero must be nowhere dense.

Using this lemma and Theorem 1.1, we obtain the following

COROLLARY 1.1. $E \in E_1 \oplus E_2$ *is conditionally poised if, and only if,* E_1 *and* E_2 *are conditionally poised.*

Proof. if E is conditionally poised, then by Theorem 1.1, so are E_1 and E_2.

If E_1 and E_2 are conditionally poised, then by Lemma 1.1 we can choose $x_1, \ldots, x_k$, with respect to which both E_1 and E_2 are poised. Theorem 1.1 yields then the fact that E is poised with respect to these points.

COROLLARY 1.2. *E is poised if, and only if,* E_1 *and* E_2 *are poised.*

We now proceed with our analysis of incidence matrices and related interpolation problems.

DEFINITION 1.3. Given an n-incidence matrix E, set $m_j = \sum_{i=1}^{k} e_{ij}$, $j = 0, 1, \ldots, n-1$, and $M_p = \sum_{j=0}^{p} m_j$ for $p = 0, \ldots, n-1$.

Notice that each m_j counts the number of ones appearing in the jth column of E and M_j counts the number of ones in columns 0 through j. For notational ease, we set $M_{-1} = 0$. We have $M_j \leqslant M_{j+1}$, $M_j + m_{j+1} = M_{j+1}$, and $M_{n-1} = n$. We call the numbers M_j *Pólya constants* since he was the first to study their importance in interpolation problems of this type.

DEFINITION 1.4. The incidence matrix E is said to satisfy the Pólya conditions if $M_j \geqslant j + 1$ for $j = 0, \ldots, n-1$.

THEOREM 1.2. *A necessary condition for E to be conditionally poised is that it satisfies the Pólya conditions.*

Proof. Suppose for some p we have $M_p \leqslant p$. Let distinct points $x_1, \ldots, x_k$ be given. Then there is a nontrivial polynomial $\tilde{p}(x) \in \pi_p$ satisfying $\tilde{p}^{(j)}(x_i) = 0$ if $e_{ij} = 1$ and $j \leqslant p$ since we have only $M_p \leqslant p$ equations and $p + 1$ parameters. But $\tilde{p}^{(p+1)}(x) \equiv 0$ and, hence, $\tilde{p}(x)$ trivially satisfies $\tilde{p}^{(j)}(x_i) = 0$ if $e_{ij} = 1$ and $j > p$. Thus, the polynomial $\tilde{p}(x)$ interpolates E at the given points. But there is no restriction on the choice of the points and we see that we can construct such an interpolating polynomial for any $x_1, x_2, \ldots, x_k$. Thus, E is not conditionally poised and the theorem is proved.

Now let us consider what happens if equality occurs in the Pólya conditions. Let $M_p = p + 1$ for some p less than $n - 1$. Define incidence matrices E_1 and E_2 by:

$$E_1 = \|e_{ij}^1\|_{j=0,\ldots,p}^{i=1,\ldots,k}, \qquad E_2 = \|e_{ij}^2\|_{j=0,\ldots,n-p-2}^{i=1,\ldots,k},$$

where

$$e_{ij}^1 = e_{ij} \qquad \text{and} \qquad e_{ij}^2 = e_{i,\,j+p+1}.$$

Then E_1 is a $(p + 1)$-incidence matrix, and E_2 is an $(n - p - 1)$-incidence matrix and $E \in E_1 \oplus E_2$. Thus, the interpolation properties of the matrix E depend solely on the interpolation properties of the matrices E_1 and E_2 as shown by Theorem 1.1 and Corollaries 1.1 and 1.2.

DEFINITION 1.5. E is said to satisfy the *strong Pólya conditions* if $M_j \geqslant j + 2$ for $j = 0, \ldots, n - 2$.

By our above remarks and Theorem 1.2, we need only consider incidence matrices which satisfy the strong Pólya conditions, since all others either reduce to lower order ones or are never poised.

2. PÓLYA SYSTEMS

We now wish to focus our attention on the case where interpolation takes place at only two nodes x_1 and x_2. That is, $k = 2$ in our incidence matrices.

There are three reasons for studying the two-point interpolation problem at this time: Practical—we need the results in Section 3, historical—it seems to be the first problem of the type we are looking at that was studied, and aesthetic—the results are particularly "nice" and complete.

Two-point systems (i.e., incidence matrices with two rows) that are poised if x_1 and x_2 are taken to be real were characterized by G. Pólya in 1931 [*4*]. Pólya's characterization was also arrived at independently by J. M. Whittaker in his book *Interpolatory Function Theory* [8]. Our result is slightly more general, as we allow the points of interpolation to assume complex values.

THEOREM 2.1. [*Pólya* 1931 *and Whittaker* 1935.] *Let E be an n-incidence matrix. If $k=2$, then E is poised (unconditionally) if, and only if, E satisfies the Pólya conditions: $M_j \geqslant j+1$ for $j=0, \ldots, n-1$. We call two-point systems which satisfy the Pólya conditions* Pólya systems.

Proof. The necessity of the Pólya conditions is shown by Theorem 1.2. We now assume that E satisfies the Pólya conditions. We first establish the fact that E is poised when x_1, x_2 are taken to be real numbers, and then we shall extend the results to the complex plane.

LEMMA 2.1. *Let E be a two-row n-incidence matrix. If x_1, x_2 are real numbers $(x_1 < x_2)$ and $p(x)$ is a real polynomial which interpolates E at x_1 and x_2, then $p^{(j)}(x)$ has at least $M_j - j$ distinct zeros on the closed interval $[x_1, x_2]$ for $j=0, \ldots, n-1$.*

Proof. E specifies $m_0 = M_0 - 0$ distinct zeros for $p(x)$ at x_1 and x_2, so the lemma is obviously true for $j=0$. Suppose that $p^{(j-1)}(x)$ has at least $M_{j-1} - (j-1)$ distinct zeros on $[x_1, x_2]$. By Rolle's Theorem, $p^{(j)}(x)$ has at least $M_{j-1} - (j-1) - 1 = M_{j-1} - j$ distinct zeros on the open interval (x_1, x_2). But E specifies m_j $(=0, 1$ or $2)$ distinct zeros of $p(x)$ at x_3 and x_2. Hence, $p^{(j)}(x)$ has at least $M_{j-1} - j + m_j = M_j - j$ zeros on $[x_1, x_2]$ and the lemma is proven by induction.

LEMMA 2.2. *If E is a two-row matrix satisfying the Pólya conditions then E is unconditionally poised whenever x_1, x_2 are real.*

Proof. By Lemma 2.1, if $p(x) \in \pi_{n-1}$ is a polynomial which interpolates E at x_3, x_2, then the constant $p^{(n-1)}(x)$ has at least $M_{n-1} - (n-1) \geqslant 1$ zeros on $[x_1, x_2]$. Hence $p^{(n-1)}(x) \equiv 0$ and $p(x) \in \pi_{n-2}$. Now, $p^{(n-2)}(x)$ has at least $M_{n-2} - (n-2) \geqslant 1$ zeros on $[x_1, x_2]$ and is itself a constant. Therefore, $p^{(n-2)}(x) \equiv 0$ and $p(x) \in \pi_{n-3}$. Continuing in this manner, we see that $p(x)$ must be a constant. But $m_0 = M_0 \geqslant 1$ and, hence, $p(x) \equiv 0$. Therefore, E is poised with respect to x_1, x_2, and the lemma is demonstrated.

To prove our theorem, we let x_1, x_2 be arbitrary points in the complex plane $(x_1 \neq x_2)$ and suppose that $p(x) \in \pi_{n-1}$ is a polynomial which interpolates E at x_1, x_2. Define $q(x) = p((x_2 - x_1)x + x_1)$. Then $q^{(j)}(0) = (x_2 - x_1)^j p^{(j)}(x_1)$, $q^{(j)}(1) = (x_2 - x_1)^j p^{(j)}(x_2)$, and we see that $q(x)$ interpolates E at $z_1 = 0$ and $z_2 = 1$. But now $q(x) \equiv 0$ by Lemma 2.2 and, hence, $p(x) \equiv 0$. Thus, E is indeed poised, and the theorem is established.

3. Conditionally Poised Systems

In this section, we shall characterize those systems which are conditionally poised in terms of the Pólya Constants M_j.

Let E be an n-incidence matrix with k rows and suppose that E satisfies the Pólya conditions, i.e., $M_j \geqslant j+1$ for each j. We shall examine the matrix E' obtained from E by suppressing its kth row. E' need not be an incidence matrix because it may have less than n nonzero entries. However, we can decompose E' into an alternating series of incidence matrices and zero matrices as we describe in the following paragraphs.

Suppose the kth row of E contains $t > 0$ ones. Suppressing this row we obtain a matrix

$$E' = \|e_{ij}\| \begin{matrix} i = 1, \ldots, k-1 \\ j = 0, \ldots, n-1 \end{matrix}$$

which contains $n-t$ ones. Let M_j' be Pólya constants for E' and choose a sequence of integers

$$-1 \leqslant j_0' < j_1' < \ldots < j_{2p}' < j_{2p+1}' \leqslant n-1 \tag{3.1}$$

satisfying

$$\left.\begin{array}{ll} M_j' = 0 & \text{if } j \leqslant j_0' \\ M_j' \geqslant j - j_0' & \text{if } j_0'+1 \leqslant j \leqslant j_1' \\ M_j' = j_1' - j_0' & \text{if } j_1'+1 \leqslant j \leqslant j_2' \\ M_j' \geqslant (j - j_2') + (j_1' - j_0') & \text{if } j_2'+1 \leqslant j \leqslant j_3' \\ M_j' = (j_3' - j_2' + (j_1' - j_0') & \text{if } j_3'+1 \leqslant j \leqslant j_4' \\ \vdots & \\ M_j' \geqslant (j - j_{2p-2}') + (j_{2p-3}' - j_{2p-4}') + \ldots + & \\ \quad + (j_3' - j_2' + (j_1' - j_0') & \text{if } j_{2p-2}'+1 \leqslant j \leqslant j_{2p-1}' \\ M_j' = (j_{2p-1}' - j_{2p-2}') + \ldots + (j_1' - j_0') & \\ & \text{if } j_{2p-1}' \leqslant j \leqslant j_{2p}' \\ M_j' \geqslant (j - j_{2p}') + (j_{2p-1}' - j_{2p-2}') + \ldots + (j_1' - j_0') & \\ & \text{if } j_{2p}'+1 \leqslant j \leqslant j_{2p+1}' \end{array}\right\} \tag{3.2}$$

and finally $M'_{j_{2p+1}} = n-t$.

In order to choose a sequence (3.1) satisfying the conditions (3.2), we proceed as follows. If the first column of E' contains a one, we let $j_0' = -1$. Otherwise we let $j_0' + 1$ be the index of the first column in E' having a one in it. Obviously, we must have $M_j' = 0$ if $j \leqslant j_0'$.

Having chosen j_0', we suppress columns 0 through j_0' of E' to obtain a new matrix. We let $j_1' + 1$ be the index in E' of the first column of the new matrix where the Pólya conditions fail. Then we have $M_j' \geqslant j - j_0'$ if $j_0' + 1 \leqslant j \leqslant j_1'$.

Note that, since this new matrix first fails to satisfy the Pólya conditions at the column labeled $j_1' + 1$ in E', we must have $m'_{j'_1+1} = 0$.

We now suppress columns 0 through j_3' of E' to obtain another new matrix. We let $j_2' + 1$ be the index in E' of the first column in the new matrix having a one. Since $m'_{j_1'+1} = 0$, we have $j_2' + 1 > j_1' + 1$ and we also have $M_j' = j_1' - j_0'$ if $j_1' + 1 \leqslant j \leqslant j_2'$. We continue in this fashion to construct a sequence (3.1) satisfying (3.2) and, by our construction, we see that the sequence is uniquely defined.

As an example, let

$$E = \begin{Vmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{Vmatrix}.$$

Then

$$E' = \begin{Vmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

and, for the sequence (3.1), we have $j_0' = 0, j_1' = 3, j_2' = 5$ and $j_3' = 7$.

Now let $k_q = j'_{2q+1} - j'_{2q}$, let E_r be the matrix consisting of columns $j'_{r-1} + 1$ through j_r' of E' and let E_{2p+2} consist of columns j'_{2p+1} to $n-1$ of E'. By our construction, we see that E_r is a zero matrix if r is even and it is a k_q-incidence matrix satisfying the Pólya conditions if $r = 2q+1$. Finally, we write $E' = E_0 + E_1 + \ldots + E_{2p+2}$. In our example, we would have

$$E_0 = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}, \quad E_1 = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad E_2 = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \quad E_3 = \begin{Vmatrix} 1 & 1 \\ 0 & 0 \end{Vmatrix}, \quad E_4 = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

We note that $k_0 + k_1 + \ldots k_p = n - t$ and the total number of columns in all the zero matrices is t.

We are now in a position to demonstrate a theorem characterizing conditionally poised systems.

THEOREM 3.1. *An n-incidence matrix*

$$E = \|e_{ij}\| \begin{matrix} i = 1, \ldots, k \\ j = 0, \ldots, n-1 \end{matrix}$$

is conditionally poised if, and only if, the Pólya conditions: $M_j \geqslant j+1$ *for* $j = 0, \ldots, n-1$ *are satisfied.*

Proof. The necessity of the Pólya conditions has been shown in Theorem 1.2. Suppose that the Pólya conditions hold and that the theorem has been demonstrated for all m-incidence matrices with $m < n$. We must now find points

$x_1, \ldots, x_k$ so that E is poised with respect to them. Suppose that the kth row of E has t ones it it. We suppress this row to obtain the matrix

$$E' = \|e_{ij}\| \begin{matrix} i = 1, \ldots, k-1 \\ j = 0, \ldots, n-1 \end{matrix}$$

and, following our remarks preceding Theorem 3.1, we write $E' = E_0 + \ldots + E_{2p+2}$ where the even-numbered matrices are zero and the odd-numbered ones are k_q-incidence matrices satisfying the Pólya conditions, the sum of the numbers k_q being $n-t$. By our inductive hypothesis and by Lemma 1.1, we can choose distinct points $x_1, \ldots, x_{k-1}$ so that each of the matrices E_{2q+1} is poised with respect to these points.

To begin our discussion, let us pick a maximal collection of linearly independent polynomials $p_1(x), \ldots, p_r(x)$ which interpolate E' at the nodes $x_1, \ldots, x_{k-1}$ so that the leading coefficient of each is one and so that their respective degrees n_i satisfy $0 \leqslant n_1 \leqslant n_2 < \ldots < n_r \leqslant n-1$. We must have $r \geqslant t$ since we have n parameters to determine and only $n-t$ equations.

Consider now the polynomial $p_i(x)$ of degree n_i. It is impossible for n_i to satisfy $j'_{2q} + 1 \leqslant n_i \leqslant j'_{2q+1}$ for any q since, if it did, the $(j'_{2q} + 1)$st derivative of $p_i(x)$ would be a nontrivial polynomial of degree less than $j'_{2q+1} - j'_{2q} = k_q$ interpolating matrix E_{2q+1} at the nodes $x_1, \ldots, x_{k-1}$ which is impossible by our choice of $x_1, \ldots, x_{k-1}$. Therefore, the only possible values for the distinct numbers $n_1, \ldots, n_r$ are the t numbers m which satisfy $j'_{2q+1} + 1 \leqslant m \leqslant {}'_{2q+2}$ for some q. Since $r \geqslant t$, this yields $r = t$, and we can set up a one-to-one correspondence between the polynomials and the columns of the zero matrices in E' by matching each such column with the polynomial having degree equal to the index of the column in E'.

In our example, we can choose $x_1 = 0$ and $x_2 = 1$ to make E_1 and E_3 poised. Then, for our polynomials, we can choose

$p_1(x) = 1$	of degree $0 = j_0'$
$p_2(x) = x^4 - (4/3)\, x^3$	of degree $4 = j_1' + 1$
$p_3(x) = x^5 - (5/3)\, x^3$	of degree $5 = j_2'$
$p_4(x) = x^8 - (8/5)\, x^5$	of degree $8 = j_3' + 1$
$p_5(x) = x^9 - (9/5)\, x^5$	of degree $9 = j_3' + 2 = n-1.$

Now, in our discussion, we see that we can choose a point x_k so that E is poised with respect to $x_1, \ldots, x_{k-1}, x_k$ if, and only if,

$$P(x) = \det\, [p_r^{(j)}(x)] \begin{matrix} r = 1, \ldots, t \\ e_{kj} = 1 \end{matrix}$$

is not identically equal to zero.

LEMMA 3.1. *Let $p_1(x), \ldots, p_t(x)$ be polynomials with leading coefficients one and of exact degrees $n_1, \ldots, n_t$, respectively. Let $0 \leqslant j_1 < \ldots < j_t$ be given integers. Then the polynomial*

$$P(x) = \det[p_i^{(j_s)}(x)]\begin{matrix} i = 1, \ldots, t \\ s = 1, \ldots, t \end{matrix}$$

is identically zero only if

$$\det[D^{(j_s)} x^{n_i}]\begin{matrix} i = 1, \ldots, t \\ s = 1, \ldots, t \end{matrix}$$

is identically zero.

Proof. $P(x)$ is a sum of terms of the form $+p_{\pi(1)}^{(j_1)}(x) p_{\pi(2)}^{(j_2)}(x) \ldots p_{\pi(t)}^{(j_t)}(x)$ where the summation is taken over all the permutations π of the sequence $1, \ldots, t$. The term of maximal degree of each summand is $\pm D^{(j_1)} x^{n_{\pi(1)}} \ldots D^{(j_t)} x^{n_{\pi(t)}}$, which is either zero or of degree $\sum_{i=1}^t n_i - \sum_{i=1}^t j_i$. Therefore, if $P(x)$ is identically zero, the sum of these terms must be zero. But their sum over all permutations of the sequence $1, \ldots, t$ is equal to

$$\det[D^{(j_s)} x^{(n_i)}]\begin{matrix} i = 1, \ldots, t \\ s = 1, \ldots, t \end{matrix}$$

and this proves the lemma.

LEMMA 3.2. If $0 \leqslant j_1 < j_2 < \ldots < j_t \leqslant n - 1$ *and the numbers n_i are increasing and satisfy $j_i \leqslant n_i \leqslant n - 1$, then*

$$\det[D^{(j_p)} x^{(n_i)}]\begin{matrix} i = 1, \ldots, t \\ p = 1, \ldots, t \end{matrix}$$

is not identically zero.

Proof. Consider the two-point interpolation problem defined by

$$\tilde{E} = \|\tilde{e}_{ij}\|\begin{matrix} i = 1, 3 \\ j = 0, \ldots, n - 1, \end{matrix}$$

where x_1 is taken to be zero,

$$\tilde{e}_{1j} = \begin{cases} 0 & \text{if } j = n_1, \ldots, n_t \\ 1 & \text{otherwise} \end{cases}, \quad \text{and} \quad \tilde{e}_{2j} = \begin{cases} 1 & \text{if } j = j_1, \ldots, j_t \\ 0 & \text{otherwise} \end{cases}.$$

The linear system corresponding to $\tilde{E}$ consists of the n equations in n unknowns given by $D^{(j)}[a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}]_{x=x_i} = 0$ if $\tilde{e}_{ij} = 1$. Now, if we look at the equations corresponding to $\tilde{e}_{1j} = 1$, we see that $a_j = 0$ if $\tilde{e}_{1j} = 1$ since $x_1 = 0$. Thus, the determinant of the linear system reduces to

$$\det[D^{(j_p)} x_2^{n_i}]\begin{matrix} i = 1, \ldots, t \\ p = 1, \ldots, t \end{matrix}.$$

Now, looking at the matrix $\tilde{E}$, we see that $\tilde{M}_j \geqslant j+1$ if $j < n_1$, where $\tilde{M}_j$ are the Pólya constants for $\tilde{E}$, since $\tilde{e}_{10} = \ldots = \tilde{e}_{1,n_1-1} = 1$. But $\tilde{e}_{2,j_1} = 1$, $j_1 \leqslant n_1$ and, hence, $\tilde{M}_{n_1} \geqslant n_3 + 1$. Also, $\tilde{e}_{1,n_1+1} = \ldots = \tilde{e}_{1,n_2-1} = 1$, which yields $\tilde{M}_j \geqslant j+1$ if $j < n_2$. But $\tilde{e}_{2,j_2} = 1$ and $j_2 \leqslant n_2$ give $\tilde{M}_{n_2} \geqslant n_2 + 1$. Continuing in this manner, we easily show that $\tilde{M}_j \geqslant j+1$ for $j = 0, \ldots, n-1$. Thus, by Theorem 2.1, $\tilde{E}$ is poised; hence

$$\det[D^{(j)} x_2^{ni}] \begin{matrix} i = 1, \ldots, t \\ e_{2j} = 1 \end{matrix}$$

is nonzero and the lemma is proven.

Now, let us return to our matrices E and E'. We see that, if the entries in the kth row of E that are one are $e_{kj_1}, \ldots, e_{kj_t}$, then we need only show that $n_s \geqslant j_s$ for each s. For if this holds, then, by Lemma 3.2,

$$\det[D^{(j_p)} x^{n_i}] \begin{matrix} i = 1, \ldots, t \\ p = 1, \ldots, t \end{matrix}$$

is nonzero and, hence, by Lemma 3.1, $P(x)$ is not identically zero. Then all we need to do is pick a point x_k, different from each of the points $x_1, \ldots, x_{k-1}$, for which $P(x_k) \neq 0$, and E will be poised with respect to the nodes $x_1, \ldots, x_{k-1}, x_k$.

LEMMA 3.3. *If E satisfies the Pólya conditions, then we must have an $n_s \geqslant j_s$ for $s = 1, \ldots, t$.*

Proof. Let n_s be the degree of one of our polynomials and let $C_0, \ldots, C_{n_s}$ be the first $n_s + 1$ columns of the matrix E'. We know that there is a q so that $j'_{2q+1} + 1 \leqslant n_s \leqslant j'_{2q+2}$. Now, the columns $C_0, \ldots, C_{n_s}$ can be divided up into those that are columns of the matrices $E_0, E_{2q} \ldots, E_{2q+2}$, and those that are columns of the incidence matrices $E_1, E_3, \ldots, E_{2q+1}$. From the relations (3.2), the total number of columns of the incidence matrices $E_1, \ldots, E_{2q+1}$ is given by $M'_{n_s} = k_1 + \ldots + k_q$, while the total number of columns belonging to the even-numbered matrices is s. Hence, $n_s + 1 = M'_{n_s} + s$, or, $M'_{n_s} = (n_s + 1) - s$.

Now $M_{n_s} = M'_{n_s} + \sum_{j=0}^{n_s} e_{kj}$. Thus, if E satisfies the Pólya conditions, then $M_{n_s} \geqslant n_s + 1$ which implies that $\sum_{j=0}^{n_s} e_{kj} \geqslant s$. But this is true if, and only if, each of the numbers j_ρ satisfies $j_\rho \leqslant n_s$ for $\rho = 1, \ldots, s$. Thus we have $j_s \leqslant n_s$, for $s = 1, \ldots, t$, and the lemma and, hence, the theorem is proved.

4. (UNCONDITIONALLY) POISED SYSTEMS

Subsection 1: Hermite Systems

We have already seen one type of poised systems in Section 2, namely the Pólya systems, where $k = 2$ and the Pólya conditions are satisfied.

We now define another class of poised systems. We say that a system E is a *Hermite system* if E has the following property: $e_{ij} = 1$ implies $e_{ij'} = 1$ for $j' \leqslant j$.

For example, the system

$$\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

is a Hermite system, as is every system describing a Newton–Lagrange interpolation, where $k = n$ and $e_{10} = \ldots = e_{n0} = 1$, and every system describing a Taylor interpolation, where $k = 1$ and $e_{10} = \ldots = e_{1,n-1} = 1$. However, a system such as

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$

is not a Hermite system.

THEOREM 4.1. *If E is a Hermite system, then E is (unconditionally) poised.*

Proof. Let E be a given Hermite system with k rows, and let $x_1, \ldots, x_k$ be distinct points. Assume, for simplicity, that E has no zero row. Since E is Hermite, we have $e_{i0} = \ldots = e_{i,\alpha_i - 1} = 1$ for each i, where $\sum_{i=1}^{k} \alpha_i = n$. But this means that any polynomial which interpolates E at the given nodes must have a zero of order at least α_i at x_i. Since $\sum_{i=1}^{k} \alpha_i = n$, if such a polynomial has degree less than n, it must be identically zero, and the theorem is proved.

Subsection 2: Two Examples

Consider the two 5-incidence matrices given by

$$E = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

and

$$\tilde{E} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{Vmatrix}.$$

What we intend to do here is to give a proof that E is unconditionally poised (although we already know that from Theorem 4.1) and that $\tilde{E}$ is not unconditionally poised, in order to illustrate the techniques we wish to develop in the remainder of the section.

Let e_{3j_1} and e_{3j_2} be the elements in the third row of E that are one. Thus, $j_1 = 0$ and $j_2 = 1$. Define sequences $I_i = (m_1{}^i, m_2{}^i)$ for $i = 1, 2$ as follows: Let $m_1{}^i$

be the column index of the first zero in the sequence $e_{ij_2}, \ldots, e_{i,n-1}$. Let $m_2{}^i$ be the column index of the first zero in the sequence $e_{ij_2}, \ldots, e_{i,n-1}$ if $m_1{}^i < j_2$. If $m_1{}^i \geqslant j_2$, let $m_2{}^i$ be the column index of the first zero in the sequence $e^i_{im_1}, \ldots, e_{i,n-1}$. Analogously, let $\tilde{e}_{3,t_1}$ and $\tilde{e}_{3,t_2}$ be the ones in the third column of $\tilde{E}$ and, in the same fashion, define the sequence I_i for M. Note that $t_1 = 0$ and $t_2 = 1$.

We have $I_1 = (2,3)$, $I_2 = (1,2)$, $\tilde{I}_1 = (1,3)$ and $\tilde{I}_2 = (0,2)$. Observe what happens if we let I be any of the four sequences and if we replace the third row of the corresponding matrix by the row defined by

$$e_{3',j} = \begin{cases} 1 & \text{if } j \in I \\ 0 & \text{otherwise} \end{cases}$$

and then allow the new third row to "coalsece" with the row corresponding to I (first row if $I = I_1$ or $I = \tilde{I}_1$, second row if $I = I_2$ or $I = \tilde{I}_2$). We get

$$E_{I_1} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}, \quad E_{I_2} = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{Vmatrix},$$

$$\tilde{E}_{\tilde{I}_1} = \begin{Vmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix}, \quad \tilde{E}_{\tilde{I}_2} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{Vmatrix},$$

and all four matrices are conditionally poised.

From Section 3, we can write

$$E' = \begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix},$$

$$\tilde{E}' = \begin{Vmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix},$$

and, if we choose $x_1 = 0$, $x_2 = 1$, then the six matrices: E_{I_i}, $\tilde{E}_{\tilde{I}_i}$ for $i = 1, 2$, and

$$\begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}, \quad \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix},$$

are all poised with respect to these points. Now, we can choose $p_1(x) = x^3 - x^2$ and $p_2(x) = x^4 - x^2$ as interpolating polynomials for E' and $q_1(x) = (1/3)x^3 - x$, $q_2(x) = (1/4)x^4 - x$ as interpolating polynomials for $\tilde{E}'$.

Let

$$P(x) = \det\begin{bmatrix} p_1(x) p_2(x) \\ p_1'(x) p_2'(x) \end{bmatrix} = x^4(x-1)^2$$

and

$$Q(x) = \det\begin{bmatrix} q_1(x) q_2(x) \\ q_1'(x) q_2'(x) \end{bmatrix} = \frac{x^3}{12}(x-1)(x^2+x-8).$$

Notice that $P(x)$ has a zero of order $4 = (m_1{}^1 - j_1) + (m_2{}^1 - j_2)$ at x_1, and a zero of order $2 = (m_1{}^2 - j_1) + (m_2{}^2 - j_2)$ at x_2. Also $Q(x)$ has a zero of order

$3 = (m_1{}^1 - t_1) + (m_2{}^1 - t_2)$ at x_1, and a zero of order $1 = (m_1{}^2 - t_1) + (m_1{}^2 - t_2)$ at x_2.

The crucial thing here is that the sequences I_i and $\tilde{I}_i$ enable us to obtain the exact order of a zero of the corresponding polynomial at x_i, and it is this property which we now wish to exploit in our characterization of unconditionally poised systems.

Now $P(x)$ has all of its zeros at 0 and 1. Hence, given any other point x, $P(x) \neq 0$ and E is poised with respect to 0, 1 and x. This is not quite a proof that E is unconditionally poised, but it is enough for our purposes here. The important thing is that $Q(x)$ does not have all its zeros at 0 and 1. Hence, there is a point x different from 0 and 1 for which $Q(x) = 0$. Thus, by the remarks in Section 3, $\tilde{E}$ is not poised with respect to the points 0, 1 and x and this shows that $\tilde{E}$ is not unconditionally poised.

Subsection 3: *The Sequences* I_i

Throughout this section, we assume that E is an n-incidence matrix with k rows and that E satisfies the Pólya conditions. We assume, further, that the kth row of E contains exactly $t > 0$ ones, given by $e_{kj_1}, \ldots, e_{kj_t}$.

For each $i = 1, \ldots, k-1$, define a sequence $I_i = (m_1{}^i, \ldots, m_t{}^i)$ as follows: Let $m_1{}^i$ be the column index of the first zero in the sequence $e_{ij_1}, \ldots, e_{i,n-1}$. Assuming that $m_1{}^i, \ldots, m_{p-1}^i$ have all been defined, where $p \leqslant t$, let $\alpha = \max(j_p, m_{p-1}^i + 1)$ and let $m_p{}^i$ be the column index for the first zero in the sequence $e_{i,\alpha}, \ldots, e_{i,n-1}$.

Before showing the existence of such a sequence for each i, let us prove the following

LEMMA 4.1. *If the sequence* I_i *exists, then it satisfies:*

(i) $0 \leqslant m_1{}^i < m_2{}^i < \ldots < m_t{}^i \leqslant n-1$;

(ii) *For each* $q, j_q \leqslant m_q{}^i$ *and* $e_{i,m_q{}^i} = 0$;

(iii) *If the sequence* $e_{ij_p}, \ldots, e_{i,m_p{}^i}$ *contains* q *zeros, then these* q *zeroes are given by* $e_{i,m^i_{p-q+1}}, e_{i,m^i_{p-q+2}}, \ldots, e_{i,m^i_p}$.

Proof. Conditions (i) and (ii) easily follow from the definition of the sequence I_i. To show that (iii) holds, we observe that if $p = 1$, then, by the definition of I_i, the sequence $e_{ij_1}, \ldots, e_{i,m_1{}^i}$ contains exactly $q = 1$ zeros given by $e_{i,m_1{}^i} = 0$, and (iii) holds.

Suppose that we have shown (iii) to hold for $m_1{}^i, \ldots, m_{p-1}^i$, and suppose that the sequence $e_{ij_p}, \ldots, e_{i,m_p{}^i}$ contains $q \geqslant 1$ zeros. If $q = 1$, then (iii) trivially, holds. If $q > 1$, let e_{ij} be the last zero in the sequence before $e_{i,m_p{}^i}$. If $j \neq m_{p-1}^i$, then $j \geqslant \max(j_p, m_{p-1}^i + 1)$ and, by definition, $j \geqslant m_p{}^i$, which is a contradiction. Therefore, $j = m_{p-1}^i$. Now, the sequence $e_{ij_{p-1}}, \ldots, e_{i,m^i_{p-1}}$ contains $q-1$ or more zeros. But, by the induction hypothesis, the last $q-1$ zeros are given by $e_{i,mi_{p-q+1}}, \ldots, e_{i,mi_{p-1}}$, and this shows that (iii) holds for p and, hence, the lemma is proved.

LEMMA 4.2. *For each $i = 1, \ldots, k-1$, the sequence I_i exists.*

Proof. If $m_1{}^i$ did not exist, then we would have $e_{ij_1} = \ldots = e_{i,n-1} = 1$. But $M_{j_1-1} \geqslant j_1$ and, since $e_{kj_1} = 1$, we must have $M_{n-1} \geqslant j_1 + 1 + (n-j_1) = n+1$ because of all the ones in the ith row. But this is impossible, so we must be able to construct $m_1{}^i$.

If $m_1{}^i, \ldots, m_{p-1}^i$ have all been constructed for $p \leqslant t$, and if $e_{i,\alpha} = \ldots = e_{i,n-1} = 1$, where $\alpha = \max(j_p, m_{p-1}^i + 1)$, let q be the last integer satisfying $m_q{}^i < j_{q+1}$ (if no such integer exists, take $q = 0$). Then, in the sequence $e_{ij_{q+1}}, \ldots, e_{i,n-1}$, there are exactly $p-(q+1)$ zeros, given by $e_{i,mi_{q+1}}, \ldots, e_{i,mi_{q-1}}$ and, hence, there are $n - j_{q+1} - p + (q+1)$ ones in the sequence. Also, we have $e_{k,j_{q+1}} = \ldots = e_{kj_t} = 1$. Thus, since $M_{j_{q+1}-1} \geqslant j_{q+1}$, we have $M_{n-1} \geqslant j_{q+1} + n - j_{q+1} - p + (q+1) + t - q = n + t - p + 1 = n+1$, which is absurd. Thus, we must be able to define $m_p{}^i$ and the lemma is proved.

Now let $S = (s_1, \ldots, s_t)$ be an increasing sequence of integers satisfying $j_q \leqslant s_q$ for each $q = 1, \ldots, t$. Fix an ith row $(1 \leqslant i \leqslant k-1)$ in E, and define a new matrix E_S by replacing the kth row of E with a row $\tilde{k}$ defined by

$$e_{\tilde{k}j} = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases}$$

and then allowing the new $\tilde{k}$th row to coalesce onto the ith row of E.

LEMMA 4.3. *E_{I_i} is an n-incidence matrix satisfying the Pólya conditions while, in general, E_S is an n-incidence matrix only if S satisfies $s_q \geqslant m_q{}^i$ for each $q = 1, \ldots, t$.*

Proof. Let $\tilde{M}_j$ denote the Pólya constants for E_{I_i} and let $I_i = (m_1, \ldots, m_t)$. We certainly have $\tilde{M}_j = M_j \geqslant j+1$ if $j \leqslant j_1 - 1$. Now $e_{ij_1} = \ldots = e_{i,m_1-1} = 1$ and this gives $\tilde{M}_j \geqslant j+1$ if $j \leqslant m_1 - 1$. But in E_{I_i}, $e_{im_1} = 1$, yielding $\tilde{M}_{m_1} \geqslant m_1 + 1$.

Suppose now that we have shown $\tilde{M}_j \geqslant j+1$ for $j \leqslant m_{p-1}$, where $p \leqslant t$. If $m_{p-1} \geqslant j_p$, then, by definition of the sequence I_i, we have $e_{im_{p-1}} = \ldots = e_{im_p} = 1$ in the matrix E_{I_i}, and this gives $\tilde{M}_j \geqslant j+1$ for $j \leqslant m_p$. If $m_{p-1} < j_p$, then $\tilde{M}_{m_{p-1}} = M_{m_{p-1}} - (p-1) + (p-1) = M_{m_{p-1}}$, since the $(p-1)$ ones in the kth row of E that are not counted by the number $\tilde{M}_{m_{p-1}}$ are compensated for by the fact that $e_{i,m_1} = \ldots = e_{i,m_{p-1}} = 1$ in E_{I_i}. Thus, we have $\tilde{M}_j = M_j \geqslant j+1$ for $j = m_{p-1}, \ldots, j_p - 1$. Now, the fact that in E_{I_i}, $e_{ij_p} = \ldots = e_{im_p} = 1$ gives $\tilde{M}_j \geqslant j+1$ for $j \leqslant m_p$.

By induction then, $\tilde{M}_j \geqslant j+1$ for $j \leqslant m_t$. But $\tilde{M}_{m_t} = M_{m_t}$. which yields $\tilde{M}_j \geqslant j+1$ for $j = 1, \ldots, n-1$, and the first part of the lemma is proved.

Now consider the matrix E_S and suppose that it is an n-incidence matrix. If $s_1 < m_1$, then, using the definition of m_1, we have $e_{is_1} = \tilde{e_{ks_1}} = 1$ and, hence, E_S would have fewer than n entries. Thus, $s_1 \geqslant m_1$.

Suppose now that we have shown $s_q \geqslant m_q$ for $q = 1, \ldots, p = 1$ where $p \leqslant t$. If $s_p < m_p$, then $m_{p-1} \leqslant s_p < m_p$. But, if $m_{p-1} = s_p$, we have $s_{p-1} = m_{p-1} = s_p$, which is impossible under the assumption that S is an increasing sequence. Now, for $m_{p-1} < s_p < m_p$, $e_{is_p} = \tilde{e_k} s_p = 1$ and E_S has fewer than n entries. Thus, if E_S is to be an n-incidence matrix, we must have $s_q \geqslant m_q$ for each q, and the lemma is proved.

Subsection 4: *The Polynomial* $P(x)$

As in Subsection 3, E is to be an n-incidence matrix satisfying the Pólya conditions. E is assumed to have k rows and the kth row has t ones in it given by $e_{kj_1}, \ldots, e_{kj_t}$. We let E' be the matrix obtained from E by suppressing its kth row, just as we did in Section 3, and we write $E' = E_0 + E_1 + \ldots + E_{2p+1} + E_{2p+2}$, where the even-numbered matrices are zero matrices, the odd-numbered ones E_{2q+1} are k_q-incidence matrices satisfying the Pólya conditions, and $\sum_{q=0}^{p} k_q = n - t$.

Let $I_1, I_2, \ldots, I_{k-1}$ be the sequences for E that were discussed in the last section and choose points $x_1, \ldots, x_{k-1}$ so that the matrices E_{2q+1} for $q = 0, \ldots, p$, and E_{I_i} for $i = 1, \ldots, k-1$ are poised with respect to these points. Construct the interpolating polynomials $p_1(x), \ldots, p_t(x)$ as in Section 3, where the degrees of the polynomials are increasing and the leading coefficient of each is 1.

Let $\mathbf{R}(x)$ represent the vector $[p_1(x), \ldots, p_t(x)]$. Define the polynomial $P(x)$ by

$$P(x) = \det [\mathbf{R}^{(j_1)}(x), \ldots, \mathbf{R}^{(j_t)}(x)].$$

We now wish to investigate this polynomial which determines whether or not there is a point x_k so that E is not poised with respect to $x_1, \ldots, x_{k-1}, x_k$.

We need the following algebraic lemma:

LEMMA 4.4.

$$P^{(r)}(x) = \sum \det [\mathbf{R}^{(j_1+r_1)}(x), \ldots, \mathbf{R}^{(j_t+r_t)}(x)],$$

where the sum is taken over all sequences $r_1, \ldots r_t$ *of nonnegative integers satisfying* $\sum_{q=1}^{t} r_q = r$ *and* $j_1 + r_1 < j_2 + r_2 < \ldots < j_t + r_t$.

Proof.

$$P'(x) = \sum_{q=1}^{t} \det [\mathbf{R}^{(j_1)}(x), \ldots, \mathbf{R}^{(j_q+1)}(x), \ldots, \mathbf{R}^{(j_t)}(x)].$$

We can delete from this sum those determinants in which $j_q + 1 = j_{q+1}$ since such a determinant has two identical rows and, hence, is zero. Thus, we can write

$$P'(x) = \sum \det [\mathbf{R}^{(j_1+r_1)}(x), \ldots, \mathbf{R}^{(j_t+r_t)}(x)],$$

where the sum is taken over all sequences $r_1, \ldots, r_t$ of nonnegative integers satisfying $\sum_{q=1}^{t} r_q = 1$ and $j_1 + r_1 < j_2 + r_2 < \ldots < j_t + r_t$.

Suppose the lemma has been demonstrated for $P^{(r-1)}(x)$. Then

$$P^{(r)}(x) = D_x\{\sum \det [\mathbf{R}^{(j_1+r_1)}(x), \ldots, \mathbf{R}^{(j_t+r_t)}(x)]\}$$

where the summation is over all sequences which add up to $r - 1$ and for which $j_1 + r_1 < \ldots < j_t + r_t$. We have

$$P^{(r)}(x) = \sum \left\{ \sum_{q=1}^{t} \det [\mathbf{R}^{(j_1+r_1)}(x), \ldots, \mathbf{R}^{(j_q+r_q+1)}(x), \ldots, \mathbf{R}^{(j_q+r_q)}(x)] \right\};$$

since we can again delete all the terms in which, for some $q, j_{q-1} + r_{q-1} + 1 = j_q$, and since $\sum_{q=1}^{t} r_q + 1 = r$, we have

$$P^{(r)}(x) = \sum \det [\mathbf{R}^{(j_1+r_1)}(x), \ldots, \mathbf{R}^{(j_t+r_t)}(x)],$$

where the summation is taken over all sequences of nonnegative integers which sum to r and for which $j_1 + r_1 < \ldots < j_t + r_t$. The lemma is, thus, demonstrated.

Now let the sequence I_i be given by $I_i = (m_1, \ldots, m_t)$ and let $m = \sum_{q=1}^{t} (m_q - j_q)$. The following important lemma tells us about the zero of $P(x)$ at the point x_i.

LEMMA 4.5. *$P(x)$ has a zero of exact order m at the point x_i.*

Proof. Suppose $r < m$. Consider the sequence $R = (j_1 + r_1, \ldots, j_t + r_t)$ in Lemma 4.4. The summand corresponding to this sequence is exactly the polynomial one would get by looking at the matrix obtained from E by replacing its kth row by the row

$$e_{\tilde{k}j} = \begin{cases} 1 & \text{if } j \in R \\ 0 & \text{otherwise.} \end{cases}$$

But then, allowing $x = x_i$, means that we are looking at the polynomial we would get by letting row $\tilde{k}$ and row i coalesce, i.e., we are looking at the linear system corresponding to the matrix E_R as in Lemma 4.3.

Now, the condition that $\sum_{q=1}^{t} r_q = r < m$ means that for some q we must have $j_q + r_q < m_q$. Using Lemma 4.3, this means that E_R is not an n-incidence matrix (in fact, E_R has fewer than n entries) and, therefore, can be interpolated by a nontrivial polynomial of degree less than n. But this implies that the linear system corresponding to it must be identically zero and, hence, the appropriate term $P^{(r)}(x_i)$ is zero. Now, this analysis holds for each term in the makeup of $P^{(r)}(x_i)$ and, hence, $P^{(r)}(x_i) = 0$.

For the same reason, each term in the expression for $P^{(m)}(x_i)$ is zero, except for the one where each $j_q + r_q = m_q$. This sequence is I_i, and the corresponding term is the polynomial for the matrix E_{I_i} at the points $x_1, \dots, x_{k-1}$. By our selection of these points, this matrix is poised with respect to them, and the term cannot be zero. Hence, $P^{(m)}(x_i) \neq 0$, and the lemma is proved.

Subsection 5: Estimating the Zeros of $P(x)$

In this subsection, we assume that E is an n-incidence matrix with k rows and satisfying the strong Pólya conditions: $M_j \geqslant j + 2$ for $j = 0, \dots, n-2$ (Def. 1.4).

Again, let the kth row of E contain t ones, let their column indices be $j_1, \dots, j_t$ and let E' be the matrix obtained from E by deleting its kth row. We write $E' = E_0 + E_1 + \dots + E_{2p+1} + E_{2p+2}$ as before. Choose points $x_1, \dots, x_{k-1}$ so that the matrices E_{2q+1} and E_{I_i} are all poised with respect to them, and form the polynomials $p_1(x), \dots, p_t(x)$ of increasing degrees $n_1 < \dots < n_t$, where the numbers n_i correspond to the column indices in E' of the columns of the matrices E_{2q}. Also, form the polynomial $P(x)$ as in the last section. We note that $M_0 = m_0 \geqslant 2$ and, hence, $j_0' = -1$ (see the beginning of Section 3) and the matrix E_0 is empty. Thus, we have $E' = E_1 + \dots + E_{2p+2}$.

We shall now prove a series of lemmas that will allow us to estimate the size of the numbers $\sum_{i=1}^{k-1} (m_q^i - j_q)$.

LEMMA 4.6. *We must have $j_1 < n_1$ and, for $q > 1$, $j_q \leqslant n_{q-1}$.*

Proof. Since E satisfies the Pólya conditions, Lemma 3.3 yields $j_q \leqslant n_q$ for each q. However, if $j_1 = n_1$, we have $M_{n_1-1} = M'_{n_1-1} = n_1$, which contradicts the assumption $M_{n_1-1} \geqslant n_1 + 1$. Hence, $j_1 < n_1$.

Consider the polynomial $p_s(x)$ of degree n_s. We know (Section 3) there is a q so that $j'_{2q+1} + 1 \leqslant n_s \leqslant j'_{2q+2}$. Now, from the relations (3.2), we have $M_{n_s} = M'_{n_s} + \sum_{j=0}^{ns} e_{kj}$ and $M'_{ns} = (m_s + 1) - s$. Suppose that we have shown $j_2 \leqslant n_1, \dots, j_s \leqslant n_{s-1}$ where $s < t$. Then we have $n_s + 2 \leqslant M_{ns} = (n_s + 1) - s + \sum_{j=0}^{ns} e_{kj}$. If $j_{s+1} > n_s$, then $\sum_{j=0}^{ns} e_{kj} = s$ and we have $n_s + 2 \leqslant n_s + 1$, which is impossible. Thus, $j_{s+1} \leqslant n_s$, and the lemma is proved.

LEMMA 4.7. *For each q and i, $m_q^i \leqslant n_q$.*

Proof. We have $j_1 < n_1$ and $e_{in_1} = 0$. Hence, we must have $m_1^i \leqslant n_1$ for each i. Suppose we have shown $m_{s-1}^i \leqslant n_{s-1}$ for each i. For any i, let $\alpha = \max(j_s, m_{s-1}^i + 1)$. Since $j_s \leqslant n_{s-1}$, $m_{s-1}^i + 1 \leqslant n_{s-1} + 1 \leqslant n_s$ and $e_{i,n_s} = 0$, we have $\alpha \leqslant n_s$ and, thus, $m_s^i \leqslant n_s$ for each i. This proves the lemma.

LEMMA 4.8. *Suppose a is a positive integer so that* $m^i_{s-a-1} < j_s - a$, *and suppose* $m^i_q \geqslant j_s$ *for some q satisfying* $s-a \leqslant q \leqslant s-1$. *Then the sequence* $e_{i,j_s-a}, \ldots, e_{i,j_s-1}$ *must contain at least* $(s-q)$ *ones.*

Proof. If the sequence in question contains fewer than $(s-q)$ ones, then it contains at least $b = a-(s-q-1)$ zeros given by $e_{i,c_1}, \ldots, e_{i,c_b}$. Now $j_{s-a} \leqslant j_s - a \leqslant e_{i,c_1}$, $m^i_{s-a-1} < e_{i,c_1}$ and, hence, $m^i_{s-a} \leqslant e_{i,c_1}$ also $j_{x-a+1} \leqslant j_s - a + 1 \leqslant e_{i,c_2}$ yields $m^i_{s-a+1} \leqslant e_{i,c_2}$. Continuing in this manner, we get $j_{s-a+b-1} = j_q \leqslant e_{i,c_b}$ and, hence, $m_q{}^i \leqslant e_{i,c_b} \leqslant j_s - 1$, which is contrary to our assumption. Thus, the lemma is proved.

Now let p_r be the number of columns in the matrix E_{2r} and let $p_0 = 0$. We have the following relationship:

$$n_{p_0+p_1+\ldots+p_{r-1}+1} + p_r - 1 = n_{p_0+\ldots+p_r}.$$

LEMMA 4.9. *Suppose column* j_s *of* E' *lies in the matrix* $E_{2q+1}+E_{2q+2}$. *Let* $\alpha_r = (s-1) - p_0 - \ldots - p_r$ *for each r. Then, either for some* $r(0 \leqslant r \leqslant q)$, *column* $j_s - \alpha_r$ *lies in the matrix* E_{2r+1}; *or, if this fails for each r, column* $j_s - \alpha_q$ *lies in* E_{2q+2}.

Proof. If $j_s - \alpha_q \geqslant n_{p_0+p_1+\ldots+p_q+1}$, then column $j_s - \alpha_q$ lies in E_{2q+2} and we are finished. Suppose now that $j_s - \alpha_q < n_{p_0+\ldots+p_q+1}$. Let r be the smallest integer so that $j_s - \alpha_r < n_{p_0+\ldots+p_r+1}$. If $r=0$, we have $j_s - (s-1) < n_1$ and, hence, column $j_s-(s-1)$ is in E_1. If $r > 0$, we have $n_{p_0+\ldots+p_{r-1}+1} \leqslant j_s - \alpha_{r-1} < j_s - \alpha_r$ since r is minimal. Adding p_r to both sides, we get $n_{p_0+\ldots+p_{r-1}+1} + p_r \leqslant j_s - \alpha_{r-1} + p_r = j_s - \alpha_r$. But $n_{p_0+\ldots+p_{r-1}+1}$ is the column index of the first column of E_{2r} and, since E_{2r} has exactly p_r columns, $n_{p_0+\ldots+p_{r-1}+1} + p_r$ is the column index of the first column of E_{2r+1}. This now gives us the fact that column $j_s - \alpha_r$ is in E_{2r+1}, and the lemma is proved.

We are now ready to establish our estimates for the numbers $\sum_{i=1}^{k+1} (m_q{}^i - j_p)$.

THEOREM 4.2. *If E satisfies the strong Pólya conditions, then we have*

$$\sum_{i=1}^{k-1} (m_1{}^i - j_1) \leqslant \begin{cases} n_1 - j_1 - 1 & \text{if } j_1 > 0 \\ n_1 - j_1 & \text{if } j_1 = 0 \end{cases}$$

and, for

$$s = 2, \ldots, t, \sum_{i=1}^{k-1} (m_s{}^i - j_s) \leqslant n_s - j_s.$$

Proof. The number $({}_1{}^i - j_1)$ counts the number of consecutive ones in row i starting with column j_1. Now this cannot exceed the number of ones in row i between columns j_1 and n_1, since $m_1{}^i \leqslant n_1$. This means that $\sum_{i=1}^{k-1} (m_1{}^i - j_1)$ is no larger than the number of ones in the matrix E_1 between columns j_1 and n_1.

Now, if $j_1 = 0$, this number is $n_1 = n_1 - j_1$. If $j_1 > 0$, then we observe that $M_j' = M_j$ for $j \leqslant j_1 - 1$, since $e_{kj} = 0$ for $j \leqslant j_1 - 1$. But this means that, in the first j_1 columns of E_1, there are at least $M'_{j_1-1} \geqslant j_1 + 1$ ones and, hence, the number of ones from the j_1st column to the n_1st column is no more than $n - (j + 1)$. Therefore, the first statement of the theorem is proved.

Suppose that column j_s lies in the matrix $E_{2q+1} + E_{2q+2}$, and that $s > 1$. From Lemma 4.9, there are two cases. To begin with, let us assume that column $j_s - \alpha_q$ lies in the matrix E_{2q+2}. Then the sequence $e_{i,\, j_s-\alpha_q}, \ldots, e_{i,\, j_s-1}, e_{ij_s}$ contains nothing but zeros. Lemma 4.7 gives $m_{s-\alpha_q-1} = m_{p_0+\ldots+p_q} \leqslant n_{p_0+\ldots+p_q} < j_s - \alpha_q$. Thus, Lemma 4.8 tells us that $m^i_{s-1} \leqslant j_s - 1$ and, hence, $m_s{}^i = j_s$. Therefore, $\sum_{i=1}^{k-1} (m_s{}^i - j_s) = 0 \leqslant n_s - j_s$.

The remaining case is that for which there is an r so that column $j_s - \alpha_r$ lies in the matrix E_{2r+1}. Then $m_{s-\alpha_r-1} = m_{p_0+\ldots+p_r} \leqslant n_{p_0+\ldots+p_r} < j_s - \alpha_r$. Now $(m_s{}^i - j_s)$ counts at most the number of ones in row i between columns j_s to n_s. Plus, it counts one for each $m_p{}^i$ that is larger than $j_s - 1$, where $p \leqslant s - 1$. But the conditions of Lemma 4.8 are met for $a = \alpha_r$, and this means that each $m_p{}^i$ that is larger than $j_s - 1$ is compensated for by a one in the sequence $e_{i,\, j_s-\alpha_r}, \ldots, e_{i,\, j_s-1}$. Thus, $(m_s{}^i - j_s)$ counts at most the number of ones in the ith row from column $j_s - \alpha_r$ to column n_s. Hence, $\sum_{i=1}^{k-1} (m_s{}^i - j_s)$ counts at most the number of ones in the matrix E' from column $j_s - \alpha_r$ to column n_s. This number is $M'_{n_s} - M'_{j_s-\alpha_r-1}$. From the relations (3.2), we have $M'_{n_s} = n_s - s + 1$. Also, we note that, if there was a one in each of the columns $n_1, \ldots, n_{p_0+\ldots+p_r}$, E' would satisfy the Pólya conditions down to the last column of E_{2r+1}. Thus, $M'_{j_s-\alpha_r-1} \geqslant j_s - \alpha_r - p_0 - \ldots - p_r$. We now have $\sum_{i=1}^{k-1} (m_s{}^i - j_s) \leqslant M'_{n_s} - M'_{j_s-\alpha_r-1} \leqslant n_s - s + 1 - j_s + \alpha_r + p_0 + \ldots + p_r = n_s - j_s$, and the theorem is proved.

Subsection 6: A Characterization of Poised Systems

We are now ready to prove our major theorem on (unconditionally) poised systems.

THEOREM 4.3. *If E satsifies the strong Pólya conditions, then E is unconditionally poised if, and only if, E is a Pólya system or a Hermite system.*

Proof. The sufficiency has already been demonstrated in Theorems 2.1 and 4.1. Suppose that E satisfies the strong Pólya conditions and that it is unconditionally poised. Then it is necessary that the polynomial $P(x)$ have no more zeros than those it has at the points $x_1, \ldots, x_{k-1}$. Now $P(x)$ has degree equal to $\sum_{q=1}^{t} n_q - \sum_{q=1}^{t} j_q$. At the point x_i, $P(x)$ has a zero of order $\sum_{q=1}^{t} (m_q{}^i - j_q)$. Hence, we must have

$$\sum_{i=1}^{k-1} \sum_{q=1}^{t} (m_q{}^i - j_q) = \sum_{q=1}^{t} n_q - \sum_{q=1}^{t} j_q$$

or

$$\sum_{q=1}^{t}\sum_{i=1}^{k-1}(m_q{}^i - j_q) = \sum_{q=1}^{t} n_q - \sum_{q=1}^{t} j_q.$$

Using Theorem 4.2, we get

$$\sum_{q=1}^{t} n_q - \sum_{q=1}^{t} j_q \leqslant \sum_{q=2}^{t}(n_q - j_q) + \begin{cases} n_1 - j_1 - 1 & \text{if } j_1 > 0 \\ n_1 - j_1 \end{cases} \quad \text{or}$$

$$0 \leqslant \begin{cases} -1 & \text{if } j_1 > 0 \\ 0 & \text{if } j_1 = 0 \end{cases}.$$

Thus, in order that E be poised, it is necessary that $j_1 = 0$ and that equality hold in each of the estimates of Theorem 4.2. Since we must have $\sum_{i=1}^{k-1} m_1{}^i = n_1$, E_1 must be a Hermite matrix.

Suppose E_1 has the only one nontrivial row. Then E must be a Pólya matrix. To see this, suppose that the nontrivial row is the first and that E_1 has a one in the second row. The columns of the matrix E_2 have indices $n_1, \ldots, n_{p_1}$. Now $j_{p_1+1} \leqslant n_{p_1}$ yields the fact that the column $j_{p_1+1} - p_1$ is in the matrix E_1. Also, to be maximal, $(m_{p_1+1}^2 - j_{p_1+1})$ must count the one in the second row of E_3. But $e_{2j} = 0$ for $j \leqslant n_{p_1}$ immediately gives $m_{p_1+1}^2 = j_{p_1+1}$ and, hence, $(m_{p_1+1}^2 - j_{p_1+1}) = 0$. Thus, if E_3 has a nontrivial row other than the first, $\sum_{i=1}^{k-1}(m_{p_1+1}^i - j_{p_1+1})$ is not maximal, and E is not poised.

Now assume that we have shown that $E_3, \ldots, E_{2q-1}$ have only one nontrivial row and that that row is the first in each of the matrices. If the second row of E_{2q+1} has a one it it, then we must have $(m_{p_1+\ldots+p_q+1}^2 - j_{p_1+\ldots+p_q+1}) \geqslant 1$. But again $m_r = j_r$ for $r \leqslant n_{p_1+\ldots+p_q}$ and, since $j_{p_1+\ldots+p_q+1} \leqslant n_{p_1+\ldots+p_q}$, we have $(m_{p_1+\ldots+p_q+1}^2 - j_{p_1+\ldots+p_q+1}) = 0 < 1$. Thus, by induction, each E_{2q+1} can have only one nontrivial row, and that row must be the same as the nontrivial row of E_1. Therefore, E has only two nontrivial rows and, hence, is a Pólya matrix.

Now, if E_1 has more than one nontrivial row, E must be a Hermite matrix. To establish this, we first show that $E' = E_1 + E_2$. If there were another matrix, E_3, with a nontrivial row (say the first), then $(m_{p_1+1}^1 - j_{p_1+1}) \geqslant 1$. But the sequence $e_{1,n_1-1}, \ldots, e_{1,n_{p_1}}$ consists entirely of zeros (if $e_{1,n_1-1} = 1$, then the fact that E_1 is Hermite implies that E can only have one nontrivial row) and, hence, $m_{p_1+1}^1 = j_{p_1+1}$. This is impossible since $(m_{p_1+1}^1 - j_{p_1+1})$ must count the one in the first row of E_3. Therefore, we must have $E' = E_1 + E_2$.

To show that E is Hermite, we now only need to show that $j_s = s - 1$ for $s = 1, \ldots, t$. We already know that $j_1 = 0$. Since $E' = E_1 + E_2$, we know that column $j_s - (s-1)$ must be in E. Also, we need the relation $\sum_{i=1}^{k-1}(m_s{}^i - j_s) = n_s - j_s$. But $\sum_{i=1}^{k-1}(m_s{}^i - j_s)$ counts at most the number of ones in E_1 from the $(j_s - s + 1)$st column on. If $j_s \geqslant s$, this number is at most $n_1 - (j_s - (s-1) + 1) = n_1 + (s-1) - j_s - 1 = n_s - j_s - 1$. Hence, we must have $j_s < s$. This gives $j_s = s - 1$, which shows that E is a Hermite system and completes the proof of the theorem.

To illustrate this theorem, let us return to the two examples of Subsection 2. Both matrices satisfy the strong Pólya conditions. The first matrix was a Hermite system and so must be unconditionally poised. The second matrix represents neither a Hermite nor a Pólya system and, hence, cannot be unconditionally poised. As a matter of fact, if we choose $x_1 = 0$ and $x_2 = 1$, then as we have shown in Subsection 2 of the present section the polynomial $P(x)$ associated with this matrix is given by $P(x) = x^3/12(x-1)(x^2+x-8)$. Now, we can choose x_3 to be either of the values $(-1 \pm \sqrt{33})/2$, so that $P(x_3) = 0$ and the system is not poised with respect to the given points. As a matter of fact, the polynomial $p(x) = q_2(x_3)q_1(x) - q_1(x_3)q_2(x)$ is a nontrivial polynomial of degree $4 < 5$ which interpolates the system at the given points.

5. Real Systems

Introduction

In Sections 3 and 4, we have characterized poised and conditionally poised interpolation systems under the assumption that the interpolation takes place in the complex plane. We now wish to analyze these systems when we restrict the interpolation to points on the real line.

Definition 5.1. An n-incidence matrix is said to be *conditionally real poised* if there are real points $x_1, \ldots, x_k$ so that E is poised with respect to them. E is said to be *(unconditionally) real poised* if it is poised with respect to every collection of real points $x_1, \ldots, x_k$.

The technique for proving theorems in this chapter will be that of counting the zeros of a possible interpolating polynomial as was done in proving that Pólya systems were poised (Theorem 2.1). The device that we shall use for this is Rolle's Theorem. We should point out here that all the polynomials that we shall consider will be assumed to be real. This is no loss of generality since, if we can interpolate a system with a nontrivial polynomial when the points are taken to be real, then the linear system for a real polynomial also has a nontrivial solution. Rolle's Theorem tells us a little more than the minimal number of zeros that the derivatives of a given polynomial must have. It also restricts their location (i.e., they must interlace with the zeros of the next lower derivative) and, for this reason, we are able to show that some systems are poised provided only that we keep the ordering of the nodes fixed. Consequently, we make the following

Definition 5.2. E is said to be *order poised* with respect to the ordering $x_1 < \ldots < x_k$ if it is poised with respect to all possible choices of the points $x_1, \ldots, x_k$ under this ordering.

If, in the proof of Theorem 3.1, we assume that all the incidence matrices are conditionally real poised, we easily obtain

THEOREM 5.1. *E is conditionally real poised if, and only if, it satisfies the Pólya conditions:* $M_j \geqslant j + 1$ *for* $j = 0, \ldots, n - 1$.

For our discussion, we need the following

DEFINITION 5.3. Let E be a given n-incidence matrix and let $p(x)$ be an interpolating polynomial for a given set of points. Let $\tilde{m}_0$ be the number of zeros of $p(x)$, including multiplicities, that are specified by E. In general, let $\tilde{m}_j$ be the number of zeros of $p^{(j)}(x)$, including multiplicities, that are specified by E but that are not counted by any of the numbers $\tilde{m}_0, \ldots, \tilde{m}_{j-1}$. Let $\tilde{M}_{-1} = 0$, and $\tilde{M}_j = \sum_{p=1}^{j} \tilde{m}_p$.

For the matrix

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

we get $\tilde{m}_j = m_j = 1$ for each j while, for the matrix

$$\begin{Vmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix},$$

we have

$$\begin{aligned} m_0 &= 2, & \tilde{m}_0 &= 3 \\ m_3 &= 2, & \tilde{m}_3 &= 2 \\ m_2 &= 1, & \tilde{m}_2 &= 0. \end{aligned}$$

The following two lemmas relate the quantities M_j and $\tilde{M}_j$.

LEMMA 5.1. *For a given* j, $M_j \geqslant j + 1$ *if, and only if,* $\tilde{M}_j \geqslant j + 1$.

Proof. Obviously, we must have $\tilde{M}_j \geqslant M_j$ for each j and, thus, we need only prove the lemma in one direction. Suppose that $\tilde{M}_j \geqslant j + 1$ for each j and that for some p we have $M_p \leqslant p$. Then, for some $j \leqslant p$, we must have $m_j = 0$ and, hence, $M_j = \tilde{M}_j$. Let j be the largest integer less than or equal to p such that $m_j = 0$. Then $M_p \leqslant p$ implies $M_j \leqslant j$ and, hence, $\tilde{M}_j = M_j \leqslant j$, which is a contradiction. Thus, the lemma holds.

LEMMA 5.2. $M_j \geqslant j + 2$ *if, and only if,* $\tilde{M}_j \geqslant j + 2$ *and* $m_0 \geqslant 2$.

Proof. Again, the proof in one direction is clear. Suppose now that $\tilde{M}_j \geqslant j + 2$ for each j, and that $m_0 \geqslant 2$. If there is a $p \geqslant 1$ such that $M_p \leqslant p + 1$, choose the smallest such p. Then, since $m_0 \geqslant 2$, we must have $m_p = 0$, which yields $\tilde{M}_p = M_p \leqslant p + 1$. This, again, is a contradiction, and the lemma is proved.

Subsection 1: *Real Poised Systems*

We know that Pólya and Hermite systems are real poised, and so are systems E where $E = E_1 + \ldots + E_p$ and each E_i is a real poised matrix. However, contrary to the complex case, these are not all the real poised systems, as the following example shows:
Let

$$E = \begin{Vmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 1 & 0 \\ 1 & 0 & \ldots & 0 & 0 \end{Vmatrix}.$$

If n is odd and x_1, x_3 ($\neq x_3$) are any two points, let $x_2 = (1/2)(x_1 + x_3)$. Then the polynomial

$$p(x) = (x - x_2)^{n-1} - \left[\frac{x_3 - x_1}{2}\right]^{n-1}$$

interpolates E at the points x_1, x_2 and x_3. However, if n is even, then E is real poised. To see this, suppose $p(x) \in \pi_{n-1}$ interpolates E at x_1, x_2 and x_3, where these points are arbitrary distinct reals. Now a real polynomial ($\not\equiv 0$) with a zero at x_1 and x_3 must have a zero of odd order for its derivative in the interval (x_1, x_3). However, according to E, $p'(x)$ has only one zero and that zero has order $n-2$, which is even. Thus, it is impossible for $p(x)$ to interpolate E, unless $p(x) \equiv 0$.

The strongest result on real poised systems that we know of, is the following

THEOREM 5.2. *For $k > 2$, suppose that the n-incidence matrix E satisfies the Pólya conditions, and suppose, further, that E has the property that each new zero for $p^{(j)}(x)(j \geqslant 1)$, specified by E, is even. That is, if $e_{i,j-1} = 0$, $e_{ij} = \ldots = e_{i,j+p-1} = 1$ and $e_{i,j+p} = 0$, then p is even. Then E is (unconditionally) real poised.*

Proof. We begin by demonstrating two lemmas.

LEMMA 5.3. *If $p(x)$, not identically zero, is a real, analytic function and $p(a) = p(b) = 0$, then $p'(x)$ has a zero of odd order in the open interval (a, b).*

Proof. $p(x)$ must have an extreme point in the interval (a, b). At this extreme point, $p'(x)$ must change sign, which implies that $p'(x)$ has a zero of odd order at this point.

LEMMA 5.4. *Let E be the n-incidence matrix given in Theorem 5.2. If $p(x)$, not identically zero, is a real, analytic function and $p^{(j)}(x_i) = 0$ if $e_{ij} = 1$, where $x_1, \ldots, x_k$ are distinct reals, then $p^{(j)}(x)$ has at least $\tilde{M}_j - j$ real zeros for $0 \leqslant j \leqslant n-1$, if we count multiplicities.*

Proof. $p(x)$ has at least $\tilde{m}_0 = \tilde{M}_0 - 0$ zeros, counting multiplicities specified by E. Suppose that the lemma has been shown for derivatives of $p(x)$ of order less than j. Thus, $p^{(j-1)}(x)$ has at least $\tilde{M}_{j-1} - (j-1)$ real zeros, including multiplicities. Rolle's Theorem now tells us that $p^{(j)}(x)$ must have at least $\tilde{M}_{j-1} - (j-1) - 1$ real zeros and these zeros are either of odd order or they are zeros of $p^{(j-1)}(x)$. However, E also specifies $\tilde{m}_j$ new zeros of $p^{(j)}(x)$ of even order. Thus, counting multiplicities, $p^{(j)}(x)$ must have at least $\tilde{M}_{j-1} - (j-1) - 1 + \tilde{m}_j$ real zeros. This gives us the fact that $p^{(j)}(x)$ has at least $\tilde{M}_j - j$ real zeros, and the lemma is proved.

Now, to prove the theorem, we suppose that $p(x) \in \pi_{n-1}$ is such that $p^{(j)}(x_i) = 0$ if $e_{ij} = 1$, where $x_1, \ldots, x_k$ are distinct real points. By Lemma 5.4, $p^{(n-1)}(x)$ has at least $\tilde{M}_{n-1} - (n-1) \geqslant 1$ zeros. But $p^{(n-1)}(x)$ is a constant. Thus, $p^{(n-1)}(x) \equiv 0$ and $p(x) \in \pi_{n-2}$.

Suppose that $p(x) \in \pi_j$, where $j \leqslant n-2$. Then, $p^{(j)}(x)$ is a constant which has at least $M_j - j \geqslant 1$ zeros, i.e. $p^{(j)}(x) \equiv 0$, and $p(x) \in \pi_{j-1}$. Since this holds for each j, we get $p(x) \in \pi_0$, namely, $p(x)$ is a constant. But $p(x)$ has at least $m_0 \geqslant 1$ zeros, which shows that $p(x) \equiv 0$ and proves the theorem.

Notice that Hermite systems are special cases of the systems described in Theorem 5.2. It would be nice to say that, if E satisfies the strong Pólya conditions, then E is real poised if, and only if, E is a Pólya system or E satisfies the conditions of Theorem 5.2. We offer this as a conjectured characterization of real, poised systems.

Subsection 2: Order Poised Systems

Referring to our example in the last subsection, we see that, whether n is even or odd, if we take $x_2 < x_1 < x_3$ or $x_1 < x_3 < x_2$, then the system is poised with respect to these points.

The first result on order poised systems that we know of, is due to Professor I. J. Schoenberg. Also K. Atkinson, A. Sharma, and J. Prasad [*2*], [*7*] have worked on such systems.

In [*5*], Professor Schoenberg discusses quasi-Hermite systems. A matrix E with k rows is said to be *quasi-Hermite* if $2 \leqslant i \leqslant k-1$ and $e_{ij} = 1$ imply $e_{ij'} = 1$ for each $j' \leqslant j$,

THEOREM 5.3. [*Schoenberg*] *If E is a quasi-Hermite matrix which satisfies the Pólya conditions, then E is order poised with respect to the ordering $x_1 < x_2 <$*

$\ldots < x_{k-1} < x_k$. *Actually, the ordering of the interior points can be completely arbitrary.*

Proof. Since E satisfies the Pólya conditions, we have $\tilde{M}_j \geqslant j+1$, by Lemma 5.1. For the purpose of proving this theorem, let m be the number of zeros, including multiplicities, that are specified by E at the points $x_2, \ldots, x_{k-1}$, and let $\tilde{m}_j$ be the number of zeros of the jth derivative, including multiplicities, specified by E at x_1 and x_k, but not previously counted. Note that the number $\tilde{m}_0$, as defined here, will usually differ from the number $\tilde{m}_0$ of Definition 5.3. However, for $j > 0$, the two definitions of $\tilde{m}_j$ agree. Also, notice that $\tilde{m}_0 + m = \tilde{M}_0$. As an example, let

$$E = \begin{Vmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{Vmatrix}.$$

Then we have $m = 3$, $\tilde{m}_0 = 0$, and the number $\tilde{m}_0$ of Definition 5.3 is 3.

LEMMA 5.5. *Under the ordering $x_1 < x_2 < \ldots < x_{k-1} < x_k$, if $p(x)$ interpolates E at $x_1, \ldots, x_k$, then $p^{(j)}(x)$ has (including multiplicities) at least $\tilde{M}_j - j$ real zeros on the interval $[x_1, x_k]$.*

Proof. $p(x)$ has $m + \tilde{m}_0 = \tilde{M}_0 - 0$ real zeros on the interval $[x_1, x_k]$. Suppose that we have shown that $p^{(j-1)}(x)$ has the required number of zeros on that interval. Then, by Rolle's Theorem, $p^{(j)}(x)$ has at least $\tilde{M}_{j-1} - (j-1) - 1$ zeros, and these zeros are either in the interior of the interval or at the end-points. But those zeros at the end-points that Rolle's Theorem guarantees must also be zeros of $p^{(j-1)}(x)$. Now, E also specifies an additional $\tilde{m}_j$ zeros for $p^{(j)}(x)$ at the end-points. Thus, $p^{(j)}(x)$ has at least $\tilde{M}_{j-1} - (j-1) - 1 + \tilde{m}_j = \tilde{M}_j - j$ zeros on the interval $[x_1, x_k]$, and the lemma is proved.

The theorem now follows in exactly the same fashion as Theorem 5.2.

Our final results on order poised systems involves those systems which we shall call pyramid systems.

DEFINITION 5.4. Let the n-incidence matrix E have k rows. Let f_i be the column index of the first one which appears in row i. E is called a *pyramid matrix* if, for each i, $e_{ij} = 1$ implies $e_{ij'} = 1$ for $f_i \leqslant j' \leqslant j$, and there is some value of i $(1 \leqslant i \leqslant k)$ so that $f_1 \geqslant f_2 \geqslant \ldots \geqslant f_i$ and $f_i \leqslant f_{i+1} \leqslant \ldots \leqslant f_k$.

As examples, the matrices

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

and

$$\begin{Vmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{Vmatrix}$$

are pyramid matrices, while the matrix

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{Vmatrix}$$

is not.

THEOREM 5.4. *If E is a pyramid matrix with k rows, satisfying the Pólya conditions, then E is poised with respect to the ordering $x_1 < \ldots < x_k$.*

Proof. To prove this theorem, we need only establish the following lemma, and then the proof follows as in Theorems 5.2 and 5.3.

LEMMA 5.6. *If $p(x)$ interpolates E at the points $x_1 < \ldots < x_k$, then $p^{(j)}(x)$ has at least $\tilde{M}_j - j$ zeros on the smallest interval containing the points x_i for which $f_i \leqslant j$.*

Proof. As usual, $p(x)$ has at least $\tilde{M}_0 - 0$ zeros at the points x_i for which $f_i = 0$. Suppose that $p^{(j-1)}(x)$ has at least $\tilde{M}_{j-1} - (j-1)$ zeros on the smallest interval containing the points x_i for which $f_i \leqslant j-1$. Then $p^{(j)}(x)$ must have at least $\tilde{M}_{j-1} - (j-1) - 1$ zeros on this interval by Rolle's Theorem. But, because of the ordering of the x_i's, none of the points for which $f_i = j$ lies in this interval and, hence, $p^{(j)}(x)$ has $\tilde{m}_j$ zeros off the interval. This now tells us that $p^{(j)}(x)$ has at least $\tilde{M}_j - j$ zeros on the smallest interval containing the points for which $f_i \leqslant j$, and the lemma is proved.

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